Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Lecture 6: Gaussian Random Variable, Other Distribution and Density Types

Department of Communication & Electronics Engineering

Instructor

Dr. Qadri Hamarsheh

Email: qhamarsheh@philadelphia.edu.jo
Website: http://www.philadelphia.edu.jo/academics/qhamarsheh

Outlines

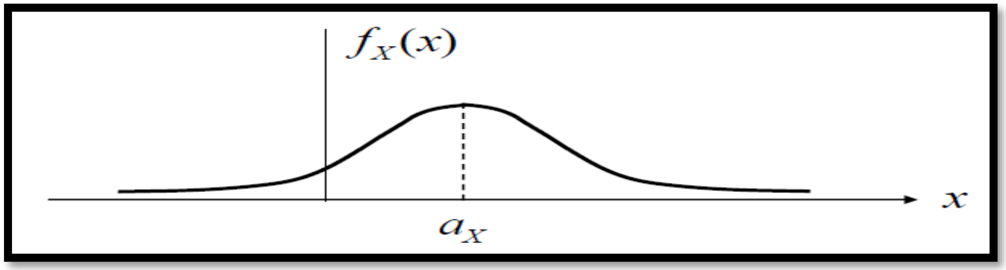
Gaussian Random Variable
Other Distribution and Density Types

Gaussian Random Variable

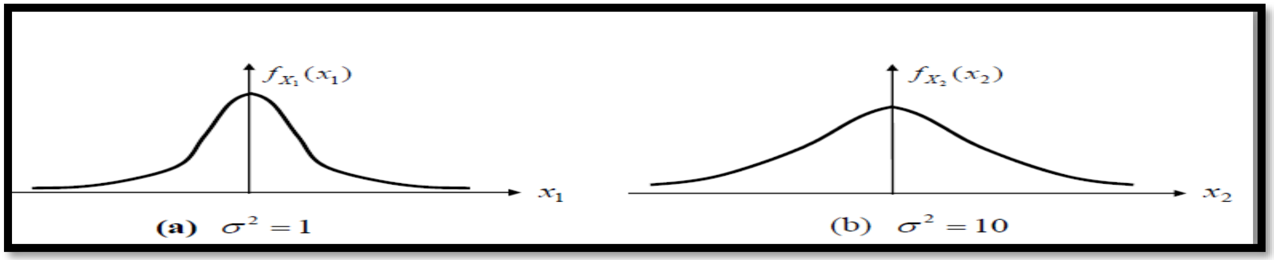
- ✓ One of the most often used **continuous probability distributions** is called the **normal probability distribution.**
- ✓ A random variable **X** is called **Gaussian** (**Normal**) if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-a_X)^2/2\sigma_X^2}$$

where $\sigma_X > 0$ and $-\infty < a_X < \infty$ are real constants.

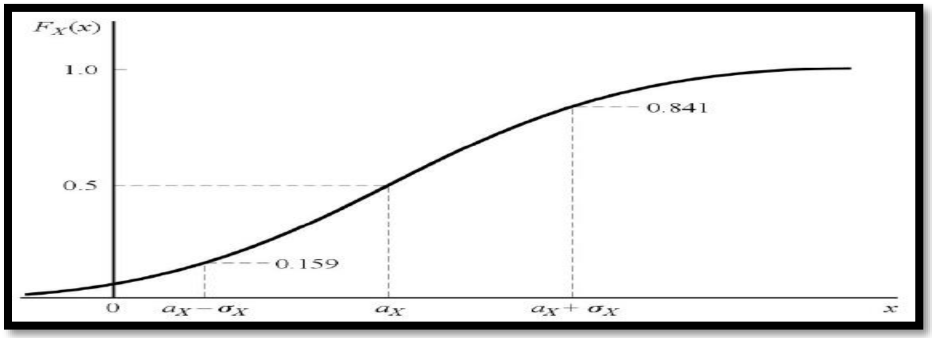


- ✓ Figure below shows two Gaussian density functions with the **same mean and different variances.** One is more concentrated around the mean, whereas the other one has a wider spread. Clearly, we need at least an additional parameter to measure this spread around the mean!



✓ The **distribution function** of the Gaussian random variable:

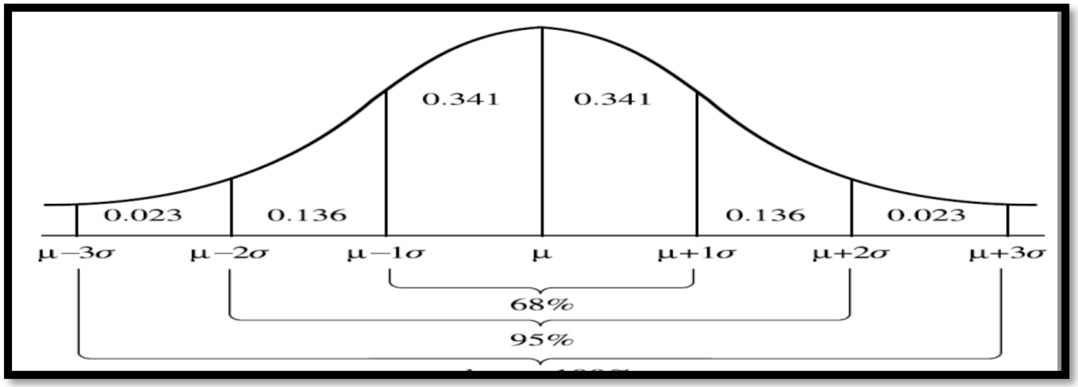
$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2 / 2\sigma_X^2} \, d\xi$$



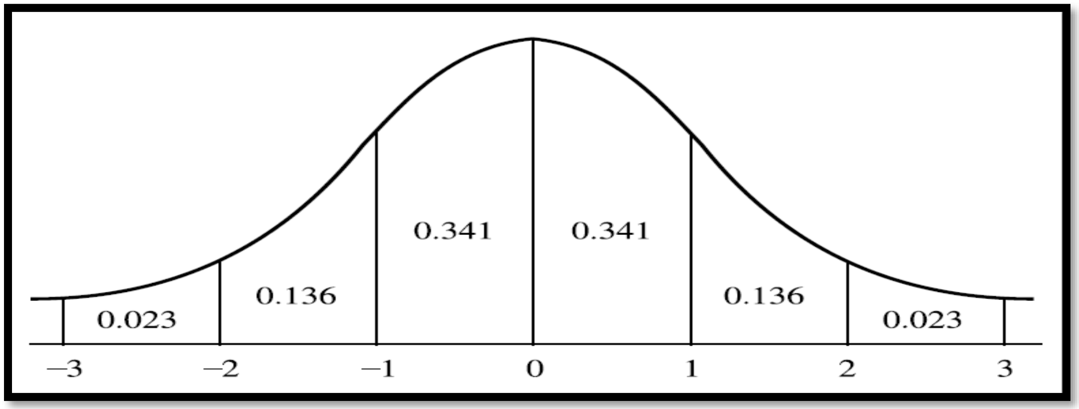
- This integral has **no known closed-form** solution and must be evaluated by numerical methods
- It appears we need a different table for every choice of values for a_X and σ_X.

✓ The normal distribution has the following properties:

1) It is **bell-shaped**.
2) The **mean**, **median**, and **mode** are at the **center** of the distribution.
3) It is **symmetric** about the **mean.**
4) It is **continuous**.
5) It **never touches** the **x** axis.
6) The **total area** under the curve is **1 or 100%.**
7) About 0.68 or 68% of the area under the curve falls within one standard deviation on either side of the mean. About 0.95 or 95% of the area under the curve falls within two standard deviations of the mean. About 1.00 or 100% of the area falls within three standard deviations of the mean.



- **Normalized Gaussian distribution function F(x)**
- The **standard normal distribution** has all the properties of a normal distribution, but the mean is zero and the standard deviation is one



- A value for any variable that is approximately normally distributed can be transformed into a standard normal value by using the following formula:

$$\boxed{z = \frac{\text{value} - \text{mean}}{\text{standard deviation}}}$$

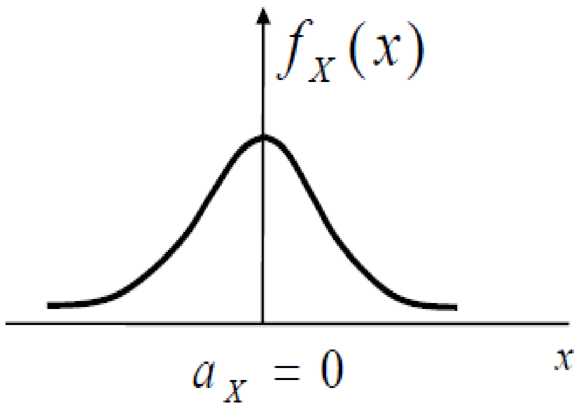
- The standard normal values are called z values or **z scores**.
- Tables for specific z values can be found in any statistics textbook

Values of (Normalized gaussian distribution function) F(x)

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} \, d\xi$$

$$\sigma_X^2 = 1 \quad \text{and} \quad a_X = 0$$



$$F(-x) = 1 - F(x)$$

✓ The general Gaussian distribution function can be found in terms of the normalized distribution function F(x) in table:

let $u = (\xi - a_X)/\sigma_X$

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-a_X)/\sigma_X} e^{-u^2/2} \, du = F\left(\frac{x - a_X}{\sigma_X}\right)$$

For example if $a_X = 1$ and $\sigma_X = 2$ then

$$F_X(3) = F\left(\frac{3-1}{2}\right) = F(1) = 0.8413$$

and

$$F_X(-3) = F\left(\frac{-3-1}{2}\right) = F(-2) = 1 - F(2)$$

$$= 1 - 0.9772 = 0.0228$$

✓ **Example**: Assume that the height of clouds above the ground at some location is a Gaussian random variable **X** with a_X=**1830m** and σ_X = **460m**. Find the probability that the clouds will be higher than **2750m**

$$\begin{aligned} P\{X > 2750\} &= 1 - P\{X \le 2750\} \\ &= 1 - F_X(2750) \\ &= 1 - F\left(\frac{2750 - 1830}{460}\right) = 1 - F(2) = 0.0228 \end{aligned}$$

Other Distribution and Density Types

1) Binomial Distribution:

- **Binomial Density Function:**

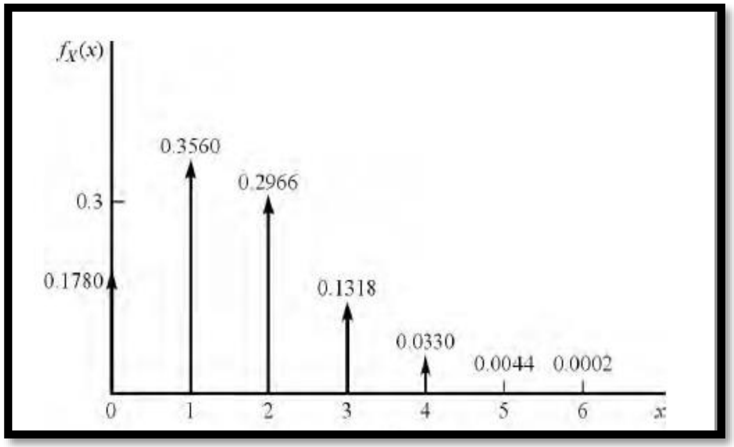
$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

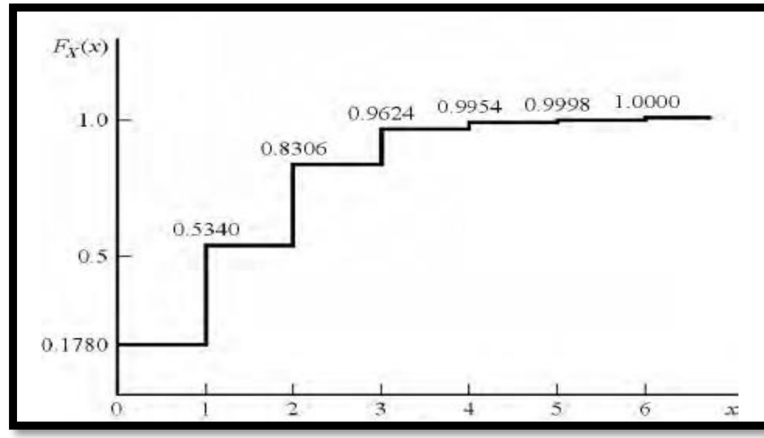
where $0 < p < 1$, and N=1,2,.... and $\binom{N}{k} = \dfrac{N!}{k!(N-k)!}$

- **The corresponding (binomial) distribution function:**

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x-k)$$

- **Example: p =0.25 and N = 6**





2) Poisson Distribution:

✓ The Poisson random variable **X** has a density and distribution given by:

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

- where $b > 0$ is a real number

✓ The probability of **x** successes is

$$\frac{e^{-\lambda}\lambda^x}{x!}$$

Where **e** is a mathematical constant $\approx$**2.7183** and λ is the mean or expected value of the variable.

- ✓ **Example**: If there are **150** typographical errors randomly distributed in a **600**-page manuscript, find the probability that any given page has exactly two errors.
  - **Solution**: Find the mean numbers of errors

$$\boxed{\lambda = \frac{150}{600} = \frac{1}{4} \text{ or } 0.25}$$

In other words, there is an average of **0.25** errors per page. In this case, **x=2**, so the probability of selecting a page with exactly two errors is

$$\boxed{\frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{(2.7183)^{-0.25} \cdot (0.25)^{2}}{2!} = 0.024}$$

Hence the probability of two errors is about **2.4%.**

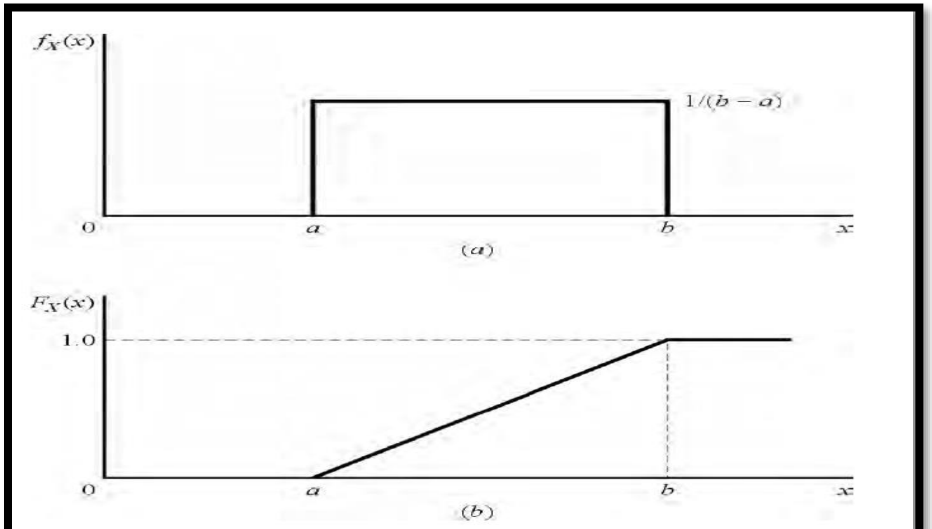
3)Uniform Distribution:

- The Uniform probability density and distribution functions are given by:

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

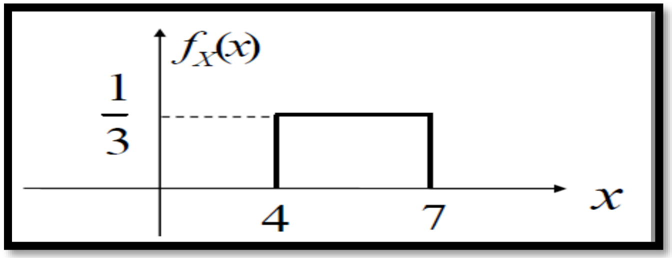
for real constants $-\infty < a < \infty$ and $b > a$.

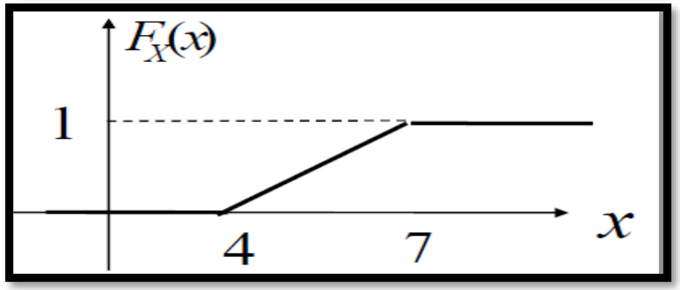


○ **Example**:

$$f_X(x) = \begin{cases} 1/3 & 4 \le x \le 7 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ (x-4)/3 & 4 \le x < 7 \\ 1 & 7 \le x \end{cases}$$





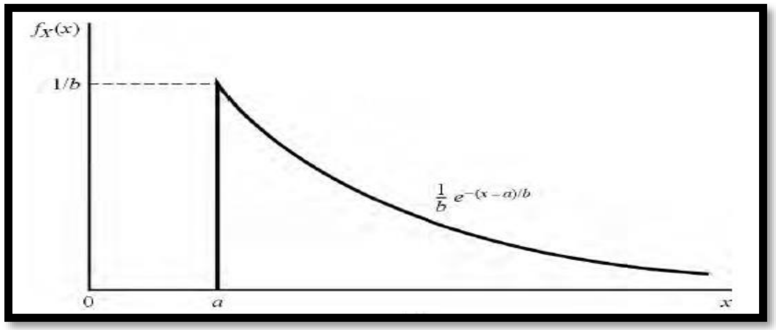
4)Exponential Distribution:

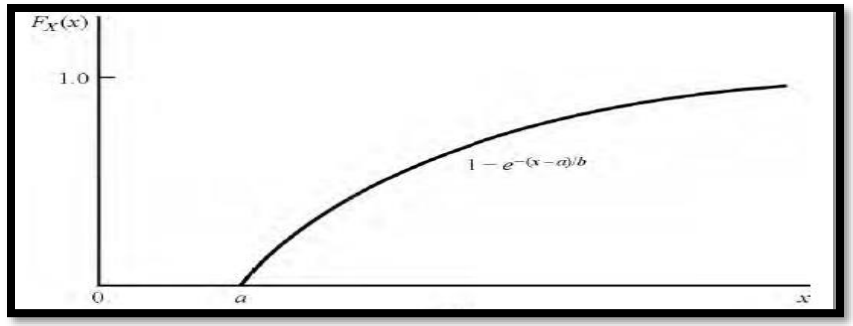
○ The exponential density and distribution functions are:

$$f_X(x) = \begin{cases} \dfrac{1}{b} e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

for real numbers $-\infty < a < \infty$ and $b > 0$.





5) Rayleigh Distribution:

- The Rayleigh density and distribution functions are:

$$f_X(x)=\begin{cases}\dfrac{2}{b}(x-a)e^{-(x-a)^2/b} & x\ge a\\ 0 & x<a\end{cases}$$

$$F_X(x)=\begin{cases}1-e^{-(x-a)^2/b} & x\ge a\\ 0 & x<a\end{cases}$$

for real numbers $-\infty<a<\infty$ and $b>0$.